Samuel D. Bellows. "Comparing the end correction of a spherically baffled piston to infinitely baffled and unbaffled circular radiators (L)". in: *J. Acoust. Soc. Am.* 155.5 (May 2024), pp. 3302–3305. DOI: 10.1121/10.0026023. URL: https://doi.org/10.1121/10.0026023

JASA LETTER TO THE EDITOR

Comparing the end correction of a spherically baffled piston to infinitely baffled and unbaffled circular radiators (L)

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ABSTRACT:
Models of acoustical systems commonly employ end corrections to represent the radiation impedance of a vibrating element. Although several analytic solutions appear in the literature, the end corrections of an infinitely baffled circular piston and an unbaffled semi-infinite circular pipe remain popular in modeling applications. This Letter compares the end correction of these two configurations to that of a radially vibrating cap on a sphere. The results show that the spherically baffled end correction, when expressed as a function of spherical-cap radius, falls between these two extremal boundary conditions. https://doi.org/10.1121/10.0026023

(Received 12 March 2024; accepted 29 April 2024; published online 16 May 2024)

[Editor: David E. Scarborough] Pages: 3302–3305

I. INTRODUCTION

Determining the acoustic impedance of a radiating component is an essential aspect of acoustical modeling, from transducer design[1,2] to physical modeling of musical instruments.[3,4] At low frequencies, a simplified representation of radiation effects follows by approximating the acoustic radiation impedance as a lumped-element acoustic mass. Researchers and practitioners commonly employ an "end correction" to characterize this acoustic mass. Most available analytic solutions to end corrections derive from straightforward boundary conditions and simple piston geometries, such as the well-known end correction for an infinitely baffled (flanged) circular piston.[5] Later works have obtained analytic solutions for other cases, such as for an unbaffled (unflanged) semi-infinite cylindrical pipe[6] or infinitely baffled rectangular or elliptical pistons.[7] However, the analytic solutions of end corrections for more complex but realistic geometries, such as sources with finite bodies, have remained ambiguous.

The end correction is crucial for precisely predicting radiation and resonant behavior. For example, Ingard's 1953 paper on acoustic resonators presented end corrections for circular to circular, circular to rectangular, and rectangular to rectangular cross sectional junctions to reduce "discrepancies between theory and experiments" that may arise when "the end correction is indiscriminately taken as the mass end correction for a plane circular piston in an infinite plane."[8] Accurately predicting a loudspeaker's electrical response requires proper estimation of radiation loading effects.[1] Likewise, physical models of wind instruments employ end corrections to account for radiation effects at tone holes and the instrument's bell.[3] Despite the significance of precisely modeling radiation effects in these and other applications, the actual value of the end correction is often unknown due to complicated source geometries and radiator element shapes. As a result, many works resort to adapting the end corrections of either an infinitely baffled or unbaffled circular piston or pipe. Better understanding of the end corrections of more realistic geometries would improve acoustical modeling in these areas.

More recently, Ref. 9 utilized the end correction for a radially vibrating cap on a rigid sphere (a "spherically baffled" piston) for use in predicting the low-frequency radiation from a radially vibrating cap on a rigid spherical shell with circular aperture. This Letter explores how the spherically baffled piston's end correction varies over spherical-cap angle. In particular, it details the relationship between its end correction and that of the two most commonly employed modeling approximations, that of the infinitely baffled circular piston and that of the unbaffled semi-infinite circular pipe. Results show that the spherically baffled result, when expressed as a function of cap radius, falls between these two limiting cases.

II. END CORRECTIONS OF RADIATING ELEMENTS

The acoustic impedance of a radiating element may be approximated as an acoustic mass at low frequencies as[1]

$$Z_A(\omega) \approx i\omega M_A = i\omega \frac{\rho_0 l}{S}, \tag{1}$$

where ω is the angular frequency, M_A is the acoustic mass (also termed the acoustic inertance), ρ_0 is the ambient density, S is the surface area of the radiating portion, and l—with units of length—is often termed as the "end correction." End corrections for many simple geometries, such as infinitely baffled elliptical and rectangular pistons, are readily available in the literature.[3,7]

Two of the most commonly employed end corrections are that of an infinitely baffled circular piston, derived by Rayleigh as[5]

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$$l_b = \frac{8}{3\pi} a_p \approx 0.8488 a_p, \tag{2}$$

and that of an unbaffled semi-infinite circular pipe, derived by Levine and Schwinger as[6]

$$l_{ub} \approx 0.6133 a_p. \tag{3}$$

In these equations, a_p is the radius of the piston or the pipe, where a "piston" refers to a vibrating component with constant and uniform surface velocity (infinite internal impedance). The termination of a "pipe" does not generally satisfy these same boundary conditions. The distinction is important because the end correction for a piston and pipe are different even for the same geometry, e.g., an infinitely baffled circular pipe takes on a value of $l \approx 0.8216 a_p$,[10] slightly lower than the value for an infinitely baffled circular piston. See Ref. 11 for further discussion.

However, most real sources are neither infinitely baffled nor completely unbaffled. One might anticipate, at least for circular pistons, that the end correction for a radiating component of a source with a finite body may take on an intermediate value. The end correction for a spherically baffled piston of cone half-angle θ_c (see Fig. 1) is given as[9]

$$l_{sb} = \frac{2a}{(1-\cos\theta_c)} \sum_{n=0}^{\infty} \frac{A_n^2}{(2n+1)(n+1)}, \tag{4}$$

where a is the sphere's radius and A_n are the expansion coefficients for a radially vibrating cap on a sphere:[1]

$$A_n = \begin{cases} \frac{1}{2}(1-\cos\theta_c), & n = 0, \\ \frac{1}{2}(P_{n-1}(\cos\theta_c) - P_{n+1}(\cos\theta_c)), & n > 0, \end{cases} \tag{5}$$

with P_n being the Legendre polynomials.

The value of the end correction divided by the sphere's radius as a function of cone half-angle θ_c appears in Fig. 2. The limiting cases occur as $\theta_c \to 0$ and $\theta_c \to \pi$. In the former, the cap surface area $S = 2\pi a^2(1-\cos\theta_c)$ vanishes so that the total acoustic mass displaced by the piston must likewise tend to 0. As a result, as $\theta_c \to 0$, the end correction $l_{sb} \to 0$.

As $\theta_c \to \pi$, the expansion coefficients become $A_n = \delta_{n0}$, with δ_{nm} the Kronecker delta, so that $l_{sb} \to a$, the sphere's radius. When $\theta_c = \pi$, the radiation is equivalent to that of a pulsating sphere, with acoustic impedance given by[1]

$$Z_A(\omega) = \frac{i\omega\rho_0 a}{S(1+ika)} \approx i\omega\frac{\rho_0 a}{S}, \quad ka \ll 1, \tag{6}$$

where k is the wavenumber and $S = 4\pi a^2$ is the sphere's surface area. By inspection, the "end correction" for the pulsating sphere is thus $l = a$, in concordance with the limiting case for Eq. (4) (for a uniformly vibrating sphere with no definite "end," the term effective acoustic mass length would be more appropriate). The value l_{sb}/a monotonically increases from zero to one as it interpolates between these two extremes.

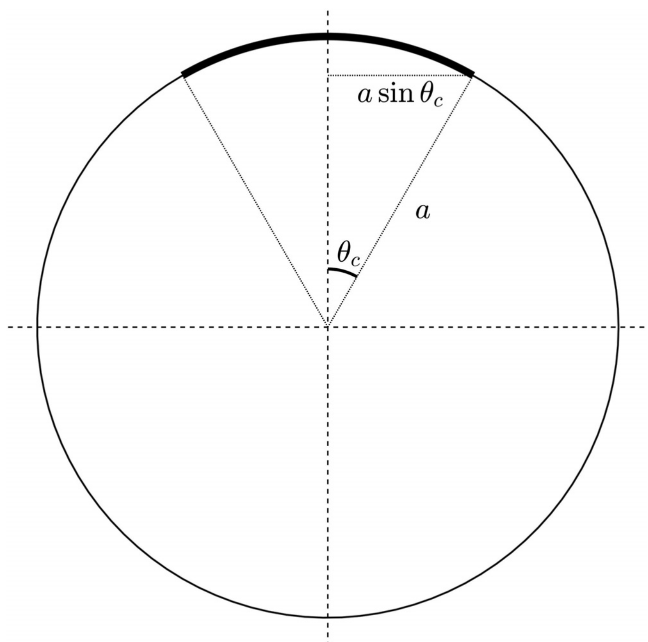

FIG. 1. Diagram of a radially vibrating cap on a sphere. The normal component of the particle velocity is constant over the spherical cap, depicted as the thick solid black curve.

III. RELATIONSHIP BETWEEN END CORRECTIONS

The end correction for the vibrating cap on a sphere ranges between 0 and a as θ_c goes from 0 to π. However, it is interesting to consider the end correction of the

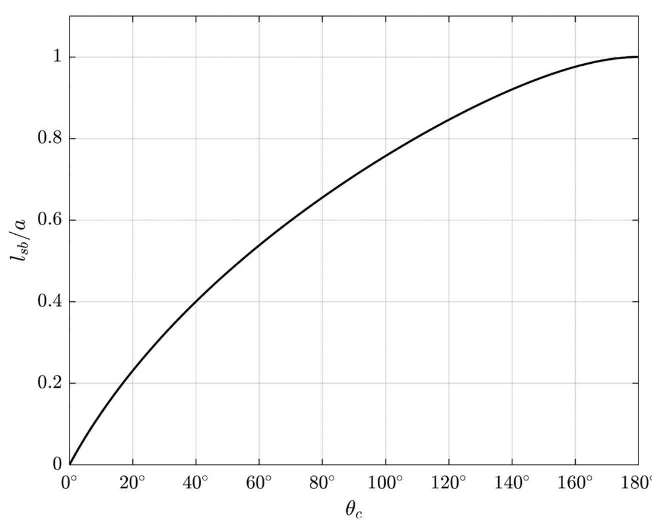

FIG. 2. End correction divided by the sphere radius a of a radially vibrating cap on a sphere as a function of cone half-angle.

spherically baffled piston as a function of the spherical-cap radius, given as (see Fig. 1)

$$a_c = a\sin\theta_c, \tag{7}$$

rather than as a function of the sphere's radius a. For the baffled and unbaffled circular piston and pipe, the end corrections take the form of Ca_p, where $C = l/a_p$ is a constant that depends on the boundary conditions. For example, $C = 8/3\pi$ for a baffled circular piston and $C \approx 0.6133$ for the unbaffled semi-infinite circular pipe. Dividing the end correction of the spherically baffled piston by the cap radius allows a more straightforward comparison between end correction values.

Of particular interest is the limit as $\theta_c \to 0$:

$$\lim_{\theta_c\to 0} l_{sb}/a_c = \lim_{\theta_c\to 0}\frac{2}{(1-\cos\theta_c)\sin\theta_c}\sum_{n=0}^{\infty}\frac{A_n^2}{(2n+1)(n+1)}. \tag{8}$$

While the end correction l_{sb} goes to zero as $\theta_c \to 0$ (see Fig. 2), so too does a_c. One anticipates that in this limit, the increasingly small spherical-cap piston may see an impedance similar to that of an infinitely baffled circular piston.

First, it is convenient to simplify the expansion coefficients A_n for very small values of θ_c. Using the large-order approximation of the Legendre polynomials yields [Eq. (14.15.11) in Ref. 12]

$$P_n(\cos\theta_c) \approx \sqrt{\frac{\theta_c}{\sin\theta_c}}J_0((n+1/2)\theta_c), \quad n \gg \cos\theta_c, \tag{9}$$

where J_0 is the Bessel function of order zero. Additionally, since $\theta_c \ll 1$, $\sin\theta_c \approx \theta_c$ so that

$$P_n(\cos\theta_c) \approx J_0((n+1/2)\theta_c). \tag{10}$$

The expansion coefficients become

$$A_n = \frac{1}{2}[J_0((n-1/2)\theta_c) - J_0((n+3/2)\theta_c)]. \tag{11}$$

However, let $\eta = (n+1/2)\theta_c$ so that

$$A_n = \frac{1}{2}[J_0(\eta-\theta_c) - J_0(\eta+\theta_c)]. \tag{12}$$

Then

$$\lim_{\theta_c\to 0}\frac{J_0(\eta+\theta_c) - J_0(\eta-\theta_c)}{2\theta_c} = J_0'(\eta) \tag{13}$$

by the definition of a derivative, yielding

$$A_n = -\theta_c J_0'(\eta) = \theta_c J_1(\eta), \quad \theta_c \ll 1, \tag{14}$$

where the last step followed by using derivative relations of the Bessel functions [Eq. (10.6.2) in Ref. 13]. Although this approximation is strictly valid for $n \gg \cos\theta_c \approx 1$, in practice, the deviations between Eqs. (5) and (14) in calculating A_n for all $n \geq 0$ was less than 1% for cone half-angle values $\theta_c < 3^\circ$ (see supplementary material in Ref. 14).

Substituting this result into Eq. (4) and using the Taylor series approximations of $\sin\theta_c \approx \theta_c$ and $(1-\cos\theta_c) \approx \theta_c^2/2$ yields

$$\lim_{\theta_c\to 0} l_{sb}/a_c = \lim_{\theta_c\to 0} 2\sum_{n=0}^{\infty}\frac{J_1^2((n+1/2)\theta_c)}{\theta_c(n+1/2)(n+1)}. \tag{15}$$

Next, consider the cone half-angle θ_c to be analogous to a discrete sampling step Δz. Then, the infinite summation over the discrete index n represents adding the summand sampled at half-integer points $\Delta z/2$, $3\Delta z/2$, etc., which is a midpoint Riemann sum with partition width Δz. Thus, in the limit of small Δz, one would have, for example,

$$\lim_{\Delta z\to 0}\sum_{n=0}^{\infty} J_1^2((n+1/2)\Delta z)\Delta z = \int_0^{\infty} J_1^2(z)dz \tag{16}$$

so that an integral replaces the infinite sum. In the summand of Eq. (15), the terms $(n+1/2)\theta_c = \eta$ become the continuous variable z in the integral. In addition, the term $(n+1)$ appearing in the denominator represents sampling at the right end point of the partition; in the limit of an infinitely small partition width Δz, the summand value may be assumed roughly constant so that $(n+1) \to \eta/\theta_c$. Thus

$$\begin{aligned}\lim_{\theta_c\to 0} l_{sb}/a_c &= \lim_{\theta_c\to 0} 2\sum_{n=0}^{\infty}\frac{J_1^2(\eta)}{\eta^2}\theta_c \\ &= 2\int_0^{\infty}\frac{J_1^2(z)}{z^2}dz.\end{aligned} \tag{17}$$

This integral is trivial to evaluate using Bessel function relations [Eqs. (10.6.1) and 10.22.57) in Ref. 13]

$$\begin{aligned}2\int_0^{\infty}\left(\frac{J_1(z)}{z}\right)^2 dz &= \int_0^{\infty}(J_0(z)+J_2(z))\left(\frac{J_1(z)}{z}\right)dz \\ &= \frac{2}{\pi}\left(1+\frac{1}{3}\right) \\ &= \frac{8}{3\pi}.\end{aligned} \tag{18}$$

Consequently,

$$\lim_{\theta_c\to 0} l_{sb}/a_c = \frac{8}{3\pi}, \tag{19}$$

which states that as the cone half-angle tends to zero, the end correction divided by the spherical-cap radius tends to that of an infinitely baffled circular piston.

The values of l_{sb}/a_c over varying cap half-angle appear in Fig. 3 for the range of $0^\circ < \theta_c < 90^\circ$ (beyond this range, and even for larger cap angles less than 90°, the use of $a_c = a\sin\theta$ is less meaningful). The overlaid dashed lines indicate the corresponding end correction values for an infinitely baffled circular piston and an unbaffled semi-infinite circular pipe.

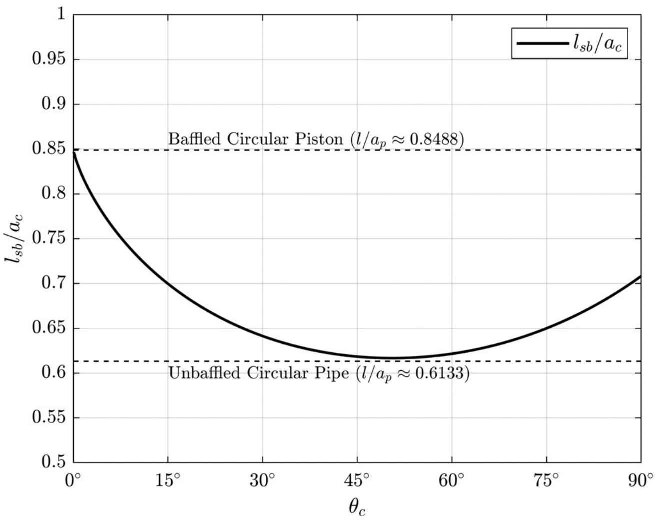

FIG. 3. End correction of a radially vibrating cap on a sphere divided by the spherical-cap radius over varying cap half-angles.

The end correction remains bounded by these two values. It obtains a maximum value as $\theta_c \to 0^\circ$ equal to that of the baffled circular piston. The spherically baffled piston's end correction approaches its infinitely baffled counterpart rapidly; it is not until $\theta_c < 3^\circ$ that the end correction is greater than $l_{sb}/a_c = 0.8$. Consequently, when applying lumped-element modeling, an infinitely baffled assumption may only be reasonable for very small pistons relative to the source's total geometry or for baffles with very little curvature.

The minimum value over this range is $l_{sb}/a_c \approx 0.6167$ at an angle of $\theta_c \approx 50.5^\circ$. This value is slightly higher than that of the unbaffled semi-infinite circular pipe, indicating that the unbaffled semi-infinite circular pipe's end correction may be a reasonable lower limit in many practical applications.

IV. CONCLUSIONS

This Letter explores the value of the end correction of a radially vibrating cap on a sphere. The end correction ranges from zero to the sphere's radius over increasing cap half-angle. The ratio of the end correction to the spherical-cap's radius remains bounded between that of an infinitely baffled circular piston and an unbaffled semi-infinite circular pipe. In particular, as the cap radius approaches zero, this ratio takes on a value corresponding to that of an infinitely baffled circular piston. The results of this work will assist in more realistically modeling the radiation impedance from sources with finite geometries.

ACKNOWLEDGMENTS

The William James and Charlene Fuhriman Strong Family Musical Acoustics Endowed Fellowship Fund partially supported the author during this work.

AUTHOR DECLARATIONS

Conflict of Interest

The author has no conflicts to disclose.

DATA AVAILABILITY

The repository at Ref. 14 contains the code used in generating the figures for this paper, including code for calculating the end correction of a radially vibrating cap on a sphere.

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